PROBABILITY

1. Basics-1

Random Experiment: Tossing A Coin Two Times

Sample Space: $S = \{HH, HT, TH, TT\}$
(Set of all possible outcomes)

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Event: $E = \{HT, TH\}$ = getting exactly one head
(A subset of S)

Impossible Event: ϕ, Empty set, Getting 3 heads
(something that can't happen)

Sure Event: S, Universal set, Getting Atmost 2 heads

Simple Event: $E_1 = \{HH\}$, $E_2 = \{HT\}$, $E_3 = \{TH\}$
$E_4 = \{TT\}$ – which has only one element

Compound Event: $E_1 = \{HT, TH\} = \{\text{exactly one head}\}$
$E_2 = \{HH, HT, TH\} = \{\text{At least one head}\}$
— A set having more than one element

Complementary Event: $E' = S - E$ = not E

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Event A or B: $A \cup B$ means either A or B or Both
(At least one of them)

Event A and B: $A \cap B$ means Both A and B

Event A but not B: $A - B = A \cap B'$ (Difference of sets)

Mutually Exclusive Events: $A \cap B = \phi$ (can't happen together)
(DISJOINT)

Exhaustive Events: $A \cup B = S$, $E \cup F \cup G = S$

M.E.+ Exhaustive: $E_1 \cup E_2 \cup E_3 \cup E_4 = S$ (All Simple Events)
and all are disjoint

2. Basics-2

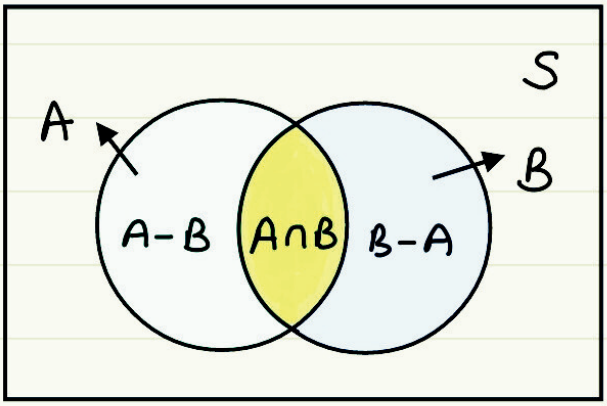
Probability of an Event, E :

$$P(E) = \frac{m}{n} = \frac{\text{No. of outcomes favourable to } E}{\text{Total possible outcomes}}$$

Equally Likely Outcomes: Events having equal Probabilities

e.g. Getting H or T in tossing a coin

$P(\phi) = 0$ (IMPOSSIBLE EVENT); $0 \leq P(E) \leq 1$; $P(S) = 1$ (SURE EVENT)



$P(A \text{ or } B) = P(A \cup B)$
$= P(\text{either } A \text{ or } B)$
$= P(\text{atleast one of } A \text{ or } B)$
$= P(A \text{ or } B \text{ or Both})$
$= P(A) + P(B) - P(A \cap B)$
$= P(A-B) + P(A \cap B) + P(B-A)$

$P(A \text{ and } B) = P(A \cap B)$
$= P(\text{Both } A \text{ and } B)$
$= P(A) + P(B) - P(A \cup B)$

$P(\text{not } A) = P(\bar{A}) = P(A') = 1 - P(A)$
$P(A) + P(\bar{A}) = 1 = P(S)$

De Morgan's Laws: $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

$P(\text{only } A) = P(A \text{ but not } B) = P(A-B) = P(A \cap B') = P(A) - P(A \cap B)$

$P(\text{only } B) = P(B \text{ but not } A) = P(B-A) = P(B \cap A') = P(B) - P(A \cap B)$

$P(\text{exactly one of } A \text{ and } B) = P(\text{only } A \text{ or only } B)$
$= P(A-B) + P(B-A) = P(A \cap B') + P(B \cap A')$
$= P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B)$

$P(\text{neither } A \text{ nor } B) = P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$
$= 1 - P(\text{either } A \text{ or } B) = P(\text{none of them})$

If A and B are disjoint (mutually exclusive)
i.e. $A \cap B = \phi$, then $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$

3. Conditional Probability

$$\boxed{P(E/F) = \frac{P(E\cap F)}{P(F)}}$$ = Prob. of E given F has already occured

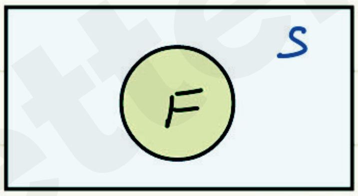
= Prob. of E taking F as the sample space

$$\boxed{P(F/E) = \frac{P(E\cap F)}{P(E)}}$$ = prob. of F given E has already occured

= Prob. of F taking E as the sample space

Properties

1. $\boxed{P(S/F) = P(F/F) = 1}$



$$P(S/F) = \frac{P(S\cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$ ($\because S\cap F = F$)

2. $\boxed{P((A\cup B)/F) = P(A/F) + P(B/F) - P((A\cap B)/F)}$

$$\text{L.H.S.} = \frac{P((A\cup B)\cap F)}{P(F)} = \frac{P(A\cap F)\cup(B\cap F)}{P(F)}$$

$$= \frac{P(A\cap F)}{P(F)} + \frac{P(B\cap F)}{P(F)} - \frac{P(A\cap B\cap F)}{P(F)} = \text{R.H.S.}$$

3. If A and B are disjoint, i.e. $A\cap B = \phi$

$\boxed{P((A\cup B)/F) = P(A/F) + P(B/F)}$ if $P(A\cap B) = 0$

4. $\boxed{P(E'/F) = 1 - P(E/F)}$ ($\because S = E\cup E'$)

$P((E\cup E')/F) = 1$ (from ① above)

$\Rightarrow P(E/F) + P(E'/F) = 1$ (using ③ above) ($\because E\cap E' = \phi$)

4. Multiplication Theorem

Multiplication Rule of Probability:

For two events :

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$$\boxed{\begin{aligned} P(E\cap F) &= P(E)\cdot P(F/E) \\ \text{or} \quad P(E\cap F) &= P(F)\cdot P(E/F) \end{aligned}}$$

$P(E)\neq 0$, $P(F)\neq 0$

$E\cap F \rightarrow$ Both E and F occur togther

Also written as EF

For three events :

$$\boxed{\begin{aligned} P(E\cap F\cap G) &= P(E)\cdot P(F/E)\cdot P(G/(E\cap F)) \\ \text{or} \quad P(EFG) &= P(E)\cdot P(F/E)\cdot P(G/EF) \end{aligned}}$$

This rule can also be used for 4 or more number of events

For Example: Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace? (NCERT)

Required Probability $= P(KKA)$

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$$= P(K).P(K/K).P(A/KK) = \frac{4}{52}\times\frac{3}{51}\times\frac{4}{50} = \frac{2}{5525}$$

5. Independent Events

Two events E and F are independent

if $P(E/F) = P(E)$

and $P(F/E) = P(F)$

$\Rightarrow \boxed{P(E \cap F) = P(E) \cdot P(F)}$

$P(E \cap F') = P(E) \cdot P(F')$

$P(E' \cap F) = P(E') \cdot P(F)$

$P(E' \cap F') = P(E') \cdot P(F')$

i.e. E & F', E' & F and E' & F' are also independent.

If $P(E \cap F) \neq P(E) \cdot P(F) \Rightarrow$ E, F are dependent

Three events A, B and C are said to be mutually independent, if

$P(A \cap B) = P(A) \cdot P(B)$ (i)

$P(A \cap C) = P(A) \cdot P(C)$ (ii)

$P(B \cap C) = P(B) \cdot P(C)$ (iii)

and $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ (iv)

All 4 ((i) to (iv)) above must be true, even if one of them is not true, the events are not independent.

Mutually Exclusive Events are defined as subsets of S which can't happen together.

Independent Events of course happen together and so they have $P(E \cap F) = P(E) \cdot P(F)$

Mutually Exclusive Events can't be Independent

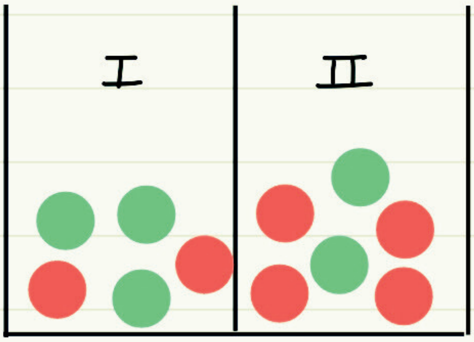
Independent Events can't be mutually Exclusive.

6. Law of Total Probability

Events: $E_1 \rightarrow$ Choosing box I

$E_2 \rightarrow$ Choosing box II

$A \rightarrow$ Drawing Red Ball



$$P(A) = P(A \cap E_1) + P(A \cap E_2)$$

$$= P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)$$

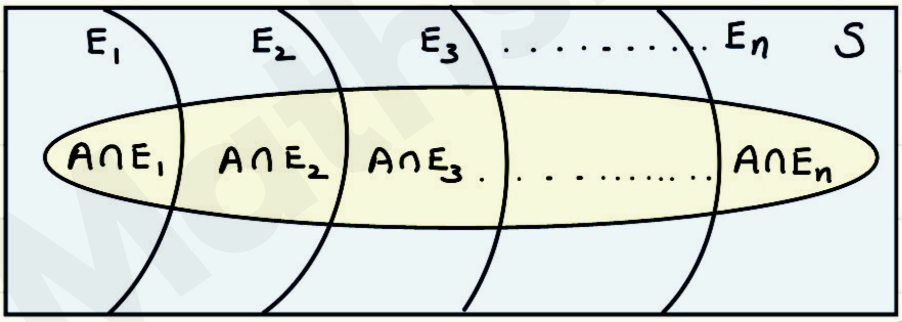
$$= \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{6} = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$$

Partition of a sample space, S:

If $E_1, E_2, E_3, \dots E_n$ are mutually exclusive and exhaustive events of S i.e.

(i) $E_i \cap E_j = \phi$, $i \neq j$, $i, j = 1, 2, 3, \dots n$

(means they are pair-wise disjoint)



(ii) $E_1 \cup E_2 \cup E_3 \dots \cup E_n = S$

(means they are exhaustive)

(iii) $P(E_i) > 0 \ \forall \ i = 1, 2, 3, \dots n$

(means all events have some non-zero probabilities)

then $P(A) = P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n)$

or $P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) \dots + P(E_n) \cdot P(A/E_n)$

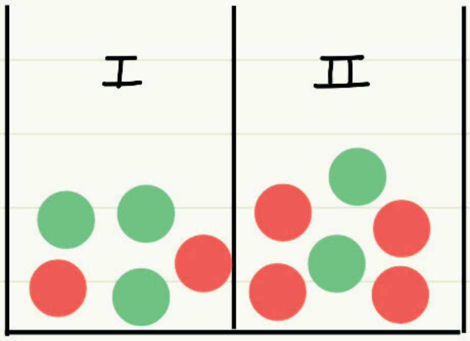
or $\boxed{P(A) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$ (Total Probability)

7. Bayes' Theorem

Events : $E_1 \rightarrow$ Choosing box I

$E_2 \rightarrow$ Choosing box II

$A \rightarrow$ Drawing Red Ball



$$P(A) = P(A \cap E_1) + P(A \cap E_2)$$

$$= P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)$$

$$= \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{6} = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$$

Given: Red Ball is drawn

To Find: Probability that it is drawn from box I

By Conditional probability, we have

$$P(E_1/A) = \frac{P(E_1 \cap A)}{P(A)}$$

$$= \frac{P(E_1 \cap A)}{P(E_1 \cap A) + P(E_2 \cap A)} \quad \left(\frac{\text{favourable}}{\text{Total}}\right)$$

$$= \frac{P(E_1)\, P(A/E_1)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{2}{5}}{\frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{6}} = \frac{3}{8}$$

In general, we have by BAYES' THEOREM:

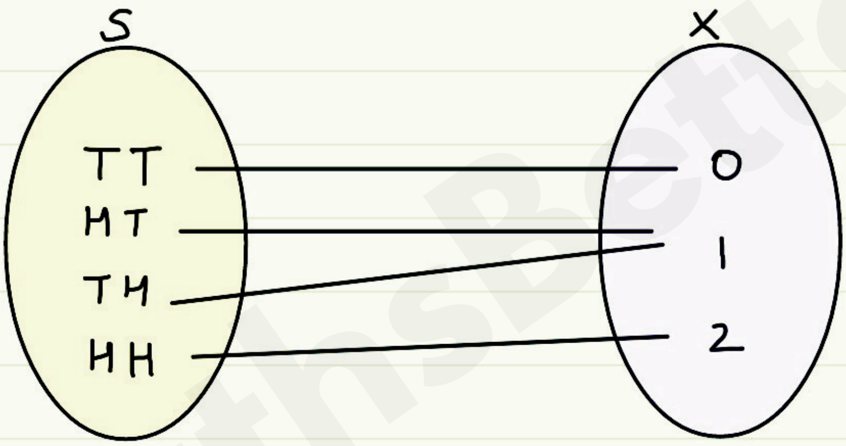
$$\boxed{P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}}, \quad i = 1, 2, \dots n$$

8. Probability Distribution

Random Variable, X (Discrete)

- → A real valued function
- → Domain = Sample Space, S
- → Range = Any Real Number

Experiment : Tossing a coin twice in succession
(or Two coins simultaneously)



Sample space   No. of Heads

$$P(X=0) = P(TT) = \frac{1}{4}$$

$$P(X=1) = P(HT, TH) = \frac{2}{4}$$

$$P(X=2) = P(HH) = \frac{1}{4}$$

$$P(X=0) + P(X=1) + P(X=2) = P(S) = 1$$

Probability Distribution Table :

X	0	1	2
P(X)	1/4	2/4	1/4

$$\sum_{i=1}^{n} p_i = 1$$

9. Mean, Variance & S.D,

If a random variable X can take the values $x_1, x_2, x_3, \dots x_n$ with their respective probabilities as $p_1, p_2, p_3, \dots p_n$, then the probability distribution of the (discrete) random variable of X is the following table :

X	x_1	x_2	x_3	...	x_n
P(X)	p_1	p_2	p_3	...	p_n

$$\sum_{i=1}^{n} p_i = 1$$

$$\left(\because \sum_{i=1}^{n} p_i = P(S)\right)$$

\# To calculate Mean of Random Variable X (or the mean of the probability distribution)

- Add one more row to the above table :

X	x_1	x_2	x_3	...	x_n
P(X)	p_1	p_2	p_3	...	p_n
X · P(X)	p_1x_1	p_2x_2	p_3x_3	...	p_nx_n

Mean: $\bar{X}$, E(X)

$$\mu = \sum_{i=1}^{n} p_i x_i$$

\# Mean is also called as mathematical Expectation or Expected value, denoted by E(x)

\# To calculate Variance and Standard Deviation

- Add one more row to the above table :

X	x_1	x_2	x_3	...	x_n
P(X)	p_1	p_2	p_3	...	p_n
X · P(X)	p_1x_1	p_2x_2	p_3x_3	...	p_nx_n
$X^2 \cdot P(X)$	$p_1x_1^2$	$p_2x_2^2$	$p_3x_3^2$	...	$p_nx_n^2$

Variance, Var(X)

$$\sigma^2 = \sum_{i=1}^{n} p_i x_i^2 - \mu^2$$

Standard Deviation, S.D.

$$\sigma = \sqrt{\text{Variance}}$$

10. Miscellaneous

Bayes' Theorem: (Formula for Probability of 'CAUSES')

Events $E_1, E_2, \ldots E_n$ are called HYPOTHESIS

Probability $P(E_i)$ is called PRIORI PROBABILITY

& Conditional Probability $P(E_i/A)$ is called a POSTERIORI PROBABILITY of the hypothesis E_i

i.e. Probability of a CAUSE (E_i) when event A has occurred

Infinite Geometric Series:

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots \infty$$

$$S_\infty = \frac{a}{1-r}, \quad |r| < 1$$

Permutations & Combinations

$${}^nP_r = \frac{n!}{(n-r)!} \qquad {}^nC_r = \frac{n!}{r!(n-r)!}$$

(for n different objects)

$${}^nP_r = r!\,{}^nC_r$$

No. of Permutations $= \dfrac{n!}{m!\,p!\,q!\ldots}$ (when all objects are not different)

F.P.C.

'AND' Rule ≡ To multiply

'OR' Rule ≡ To add

Shortcut to solve: ${}^nC_r$ is

$\frac{n}{r} \times \frac{n-1}{r-1} \times \frac{n-2}{r-2} \times \ldots$ till you get 1 in the denominator

e.g. ${}^{10}C_3 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1}$ And if it is ${}^{10}C_7$, first write it $= {}^{10}C_3$ $(\because {}^nC_r = {}^nC_{n-r})$ and then solve it.

Binomial Theorem:

$$(a+b)^n = {}^nC_0\, a^{n-0} b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_n\, a^0 b^n$$

$$= \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$$