

DIFFERENTIAL EQUATIONS

1. Basics

An equation having



- Independent Variable and/or
- Dependent Variable and
- Derivatives

is called a differential equation.

e.g. $xy' + y = 0$
 $y' = e^x$

whereas eqns. like

$$x^2 - 5x + 6 = 0$$

$$\sin x + \cos x = 0$$

are not differential equations
because they don't have derivative(s)

of a dependent variable (y) w.r.t.

an independent variable (x)



Notations:

$$\frac{dy}{dx} = y' = y_1, \quad \frac{d^2y}{dx^2} = y'' = y_2$$

$$\frac{d^3y}{dx^3} = y''' = y_3 \quad \text{In general, } \frac{d^ny}{dx^n} = y_n$$

2. Order and Degree



Order :

Order of the highest order derivative

Degree :



Highest power of the highest Order derivative

Both, Order and Degree of a differential Equation, are always positive integers

Degree is defined only if the differential Equation is a polynomial in y'

To determine degree (if defined), we must make the differential coefficients free from negative, radicals and/or fractional indices, if present, in the given differential Equation.



For example:

Differential Equation	Order	Degree
$(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$	3	2
$y_4 + \sin(y_3) = 0$	4	Not Defined
$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/4} + x^{1/5} = 0$ or $\frac{dy}{dx} = \left(\frac{d^2y}{dx^2} + x^{1/5}\right)^4$	2	4

3. Formation of Diff. Eq.

Form the differential equation representing the family of curves, $y = A \cos x + B \sin x$, — (1) where A and B are two arbitrary constants (parameters)



Steps: (Algorithm)

1. Differentiate eq. (1) two times (i.e. equal to the no. of constants)
2. Eliminate A and B
3. Equation so obtained is the required differential equation.

Sol. Differentiating (1), we get

$$y' = -A \sin x + B \cos x \text{ — (2)}$$

Differentiating (2), we get

$$\begin{aligned} y'' &= A \cos x - B \sin x \\ &= -(A \cos x + B \sin x) \text{ — (3)} \end{aligned}$$

$$y'' = -y \text{ (Eliminating } A \text{ and } B)$$

$\Rightarrow y'' + y = 0$ is the required differential equation.



Family of circles with centre at origin is:

$$x^2 + y^2 = r^2 \rightarrow \text{Corresponding differential}$$

$$\text{Equation is: } 2x + 2yy' = 0 \text{ i.e. } y' = -\frac{x}{y}$$

4. Solution of Diff. Eqn.

Verify that $y = A \cos x + B \sin x$ — (1)

is a general solution of $\frac{d^2y}{dx^2} + y = 0$ — (2)

Sol. Differentiate (1), we get

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$$\frac{dy}{dx} = -A \sin x + B \cos x$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -A \cos x - B \sin x \quad \text{--- (3)}$$

Put values of $\frac{d^2y}{dx^2}$ and y from (1) & (3) in eq. (2) we get

$$\begin{aligned} \text{L.H.S.} &= -A \cos x - B \sin x + A \cos x + B \sin x \\ &= 0 = \text{R.H.S.} \end{aligned}$$

⇒ Eq. (1) i.e., $y = A \cos x + B \sin x$ is a general sol. of the given differential equation (2).

No. of arbitrary constants in the general solution = Order of differential equation  Maths Better

Need to differentiate y as many times as the number of constants (parameters)

$y = 3 \cos x + 2 \sin x$ is a particular solution

No. of arbitrary constants in a particular solution = ZERO

5. Variable Separable

SOLVING FIRST ORDER, FIRST DEGREE D.E.

Type of diff. eq. : $\frac{dy}{dx} = F(x, y)$

or $\frac{dy}{dx} = g(x) \cdot h(y)$



$$\Rightarrow \frac{1}{h(y)} dy = g(x) dx \quad - (1)$$

Integrating both sides after separating the variables

General solution : $\int \frac{1}{h(y)} dy = \int g(x) dx + C \quad - (2)$

No need to add Constant of integration on both sides

We generally keep function of y on L.H.S. and add C on R.H.S. i.e. with function of x .

Particular Solution : $\int \frac{1}{h(y)} dy = \int g(x) dx + 2 \text{ (say)}$

e.g. Particular sol. of $y' = -4xy^2$ is :

$$y = \frac{1}{2x^2 + 1} \quad \text{when } y(0) = 1$$



Get the value of C by putting the given values of x and y in (2)

Such problems of finding a particular sol. are called INITIAL VALUE PROBLEMS.

6. Homogeneous Function



A function $F(x, y)$ is said to be a homogeneous function of degree n



if it can be expressed in the form of

(A) $x^n f\left(\frac{y}{x}\right)$ or $y^n g\left(\frac{x}{y}\right)$ Or if

(B) $F(\lambda x, \lambda y) = \lambda^n F(x, y)$, $\lambda \neq 0$

Consider, For example,

① $F(x, y) = y^2 + 2xy$

Here $F(x, y) = y^2 \left(1 + 2\frac{x}{y}\right) = y^2 g\left(\frac{x}{y}\right)$ — (A)
where $g\left(\frac{x}{y}\right) = 1 + 2\left(\frac{x}{y}\right)$

or $F(x, y) = x^2 \left(\frac{y^2}{x^2} + 2\frac{y}{x}\right) = x^2 f\left(\frac{y}{x}\right)$ — (A)
where $f\left(\frac{y}{x}\right) = \frac{y^2}{x^2} + 2\frac{y}{x}$

or $F(\lambda x, \lambda y) = \lambda^2 y^2 + 2\lambda x \cdot \lambda y = \lambda^2 (y^2 + 2xy)$

i.e. $F(\lambda x, \lambda y) = \lambda^2 F(x, y)$ — (B)

So $F(x, y)$ is a homogeneous function of degree 2

② $F(x, y) = \sin x + \cos y$



We can't make it a function of $\frac{y}{x}$ or $\frac{x}{y}$

And also $F(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq F(x, y)$

So it is not a homogeneous function.

③ $F(x, y) = \frac{x+2y}{x-y} = \frac{1+2(y/x)}{1-(y/x)} = x^0 f\left(\frac{y}{x}\right)$

Also $F(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \frac{x + 2y}{x - y} = \lambda^0 F(x, y)$

$\therefore F(x, y)$ is a homogeneous fn. of degree 0.

7. Homogeneous Diff. Eq.

TO IDENTIFY AND SOLVE HOMOGENEOUS D.E.

$$\text{Type of diff eqn.: } \frac{dy}{dx} = F(x, y) \quad - (1)$$

where $F(x, y)$ is a homogeneous fn. of degree 0
(page 6)

i.e. $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$



or $F(x, y) = \frac{g(x, y)}{h(x, y)}$, (degree of $g(x, y)$ = degree of $h(x, y)$)

or $F(x, y) = x^0 g\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right)$

Eq. (1) is a homogeneous D.E. of first order first degree

Substitute

$$y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

in (1) and proceed as in variable separable (page 5)



Similarly for

$$\text{Type: } \frac{dx}{dy} = F(x, y) \quad - (2)$$

Here homogeneous fn, $F(x, y)$ can be expressed as a function of $\left(\frac{x}{y}\right)$ i.e. $y^0 h\left(\frac{x}{y}\right)$ or $\frac{h(x, y)}{g(x, y)}$

Substitute

$$x = vy \Rightarrow \frac{dx}{dy} = v + y \frac{dv}{dy}$$

in (2) and proceed accordingly.

8. Linear Differential Eq.

TO IDENTIFY AND SOLVE FIRST ORDER, LINEAR D.E.

TYPE I : $\frac{dy}{dx} + Py = Q$ — (1)

P & Q are functions of x or constants

I.F. = $e^{\int P dx}$ (Integrating Factor)



Solution : $y(I.F.) = \int Q(I.F.) dx + C$

TYPE II : $\frac{dx}{dy} + Px = Q$ — (2)

P & Q are functions of y or constants

I.F. = $e^{\int P dy}$ (Integrating Factor)

Solution : $x(I.F.) = \int Q(I.F.) dy + C$

For example : Solve $x \frac{dy}{dx} - y = 2x^2$

Divide by x , we get $\frac{dy}{dx} - \frac{y}{x} = 2x$ (Type I)

Here, $P = -\frac{1}{x}$, $Q = 2x$

I.F. = $e^{\int -\frac{1}{x} dx} = e^{-\log(x)} = e^{\log(\frac{1}{x})} = \frac{1}{x}$

Sol. : $y \cdot \frac{1}{x} = \int 2x \cdot \frac{1}{x} dx + C \Rightarrow \frac{y}{x} = 2x + C$

or $y = 2x^2 + Cx$ (General solution)

Remember : $e^{\log_e f} = f$ (f can be function of x or y etc.)

9. Word Problems

Meaning of Differential Equations:

A differential equation (D.E.) represents a family of curves (infinite)

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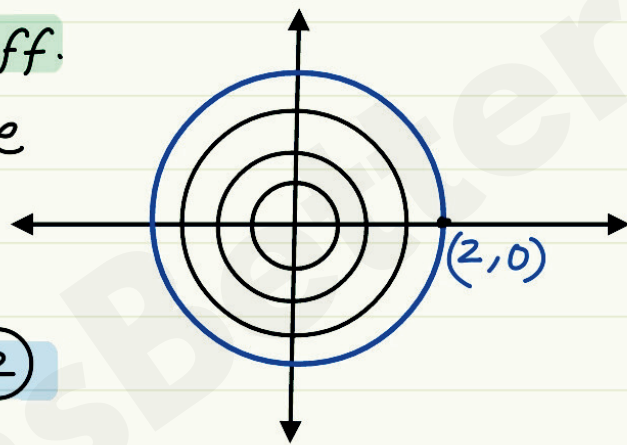
Eq. of all the circles with centre at origin

can be given by: $x^2 + y^2 = r^2$ — (1)

The corresponding diff. eqn. for (1) will be

$$2x + 2yy' = 0$$

$$\text{or } y' = -\frac{x}{y} \text{ — (2)}$$



Now (1) represents the general solution of (2) which can be easily obtained (back) by integrating eq. (2) (by variable separable) i.e.

$$y dy = -x dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\text{or } x^2 + y^2 = 2C \text{ — (3)}$$

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(where $2C$ can be written as r^2)

Now a particular sol. of (2), let's say, a circle passing through (2,0), will be:

$$2^2 + 0^2 = 2C = 4 \Rightarrow x^2 + y^2 = 4 \text{ — (4)}$$

NCERT Example 18. Find the equation of a curve passing through the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.

(MUST TRY)

10. Miscellaneous



Ordinary Differential Equation:

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- A diff. eqn. involving derivatives of the dependent variable w.r.t. only one independent variable

D.E. having more than one independent variables are called Partial Diff. Eqns. (Not in class 12)

Linear Differential Equations:

- A diff eqn. where dependent variable and its derivatives occur only in first degree and are not multiplied together. e.g. $\frac{dy}{dx} + Py = Q$

Otherwise, it is a non-linear diff. eqn. if (atleast one is true)

- Its degree is more than one.
- Any of the diff. coefficient (derivatives) has exponent more than one.
- Exponent of the dependent variable is more than one.
- There are terms containing products of dependent variable and its derivatives.

e.g. $y'' + (y')^2 + xy = 0$

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Must practice - NCERT Misch. Q.8 (looks like homogeneous, but it's not; even then can be solved by putting $x = vy$ and variable separable)

- and Misch. Q.9 (Put $x-y = t$ and reduce it to variable separable)

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