3D GEOMETRY

1. Basics

DIRECTION RATIOS D.R.'s $<a, b, c>$

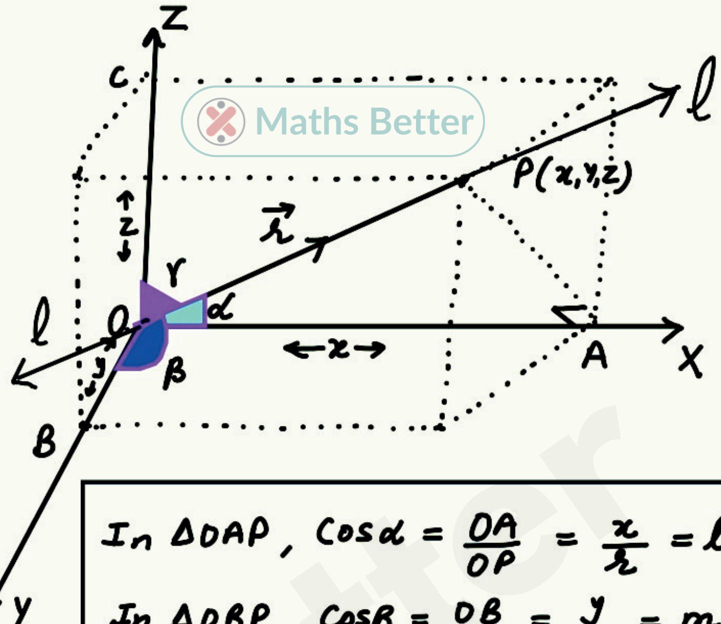
&

DIRECTION COSINES D.C.'s $<l, m, n>$

OF A LINE

are proportional

i.e. $a = \lambda l,\ b = \lambda m,\ c = \lambda n$



In ΔOAP, $\cos\alpha = \dfrac{OA}{OP} = \dfrac{x}{r} = l$

In ΔOBP, $\cos\beta = \dfrac{OB}{OP} = \dfrac{y}{r} = m$

In ΔOCP, $\cos\gamma = \dfrac{OC}{OP} = \dfrac{z}{r} = n$

$$\boxed{l^2 + m^2 + n^2 = 1}$$

or $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

α, β, γ : DIRECTION ANGLES : $\pi-\alpha, \pi-\beta, \pi-\gamma$

l, m, n : DIRECTION COSINES (d.c.'s)

a, b, c : DIRECTION RATIOS (d.r.'s)

$\dfrac{l}{a} = k,\ \dfrac{m}{b} = k,\ \dfrac{n}{c} = k$ (some constant)

$l = ak,\ m = bk,\ n = ck$

$$l = \frac{\pm a}{\sqrt{a^2+b^2+c^2}}\ ;\ m = \frac{\pm b}{\sqrt{a^2+b^2+c^2}}\ ;\ n = \frac{\pm c}{\sqrt{a^2+b^2+c^2}}$$

where $\dfrac{\pm 1}{\sqrt{a^2+b^2+c^2}} = k$

D.C.'s of 3 axes are : $<1, 0, 0>, <0, 1, 0>, <0, 0, 1>$

2. Equations of a Line

① Line thru One - Point and Parallel to a Vector (or line)

(Point-Parallel Form)

- Vector Form

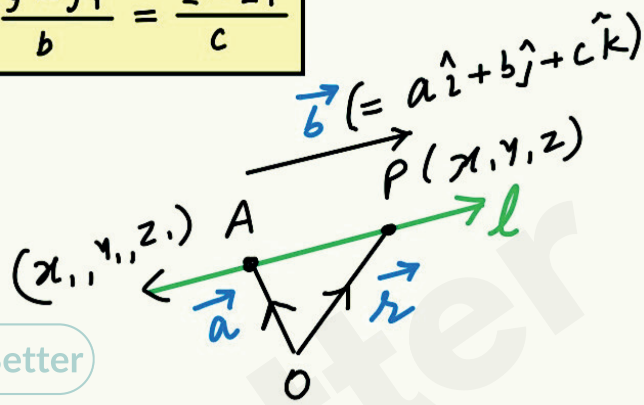
$$\vec{r} = \vec{a} + \lambda\vec{b}$$

- Cartesian Form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

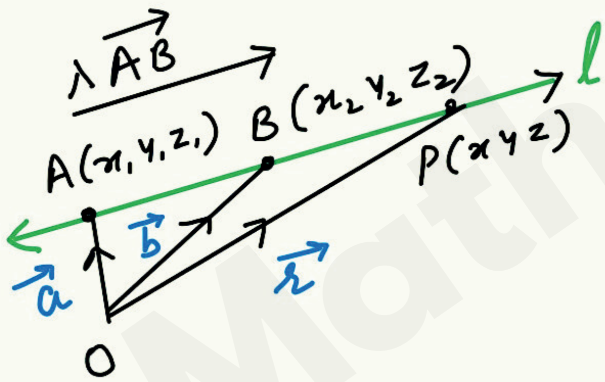
- Parametric Form

$$x = x_1 + a\lambda$$
$$y = y_1 + b\lambda$$
$$z = z_1 + c\lambda$$



Maths Better

② Line thru Two - Points (Two-Point Form)



- Vector Form

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

- Cartesian Form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

- 3 points A, B & C are collinear if $\vec{AB} = \lambda\vec{BC}$ or $\mu\vec{AC}$ etc.

③ Line thru a point and Perpendicular to Two Lines

(Point-Perpendicular Form)

- Vector Form

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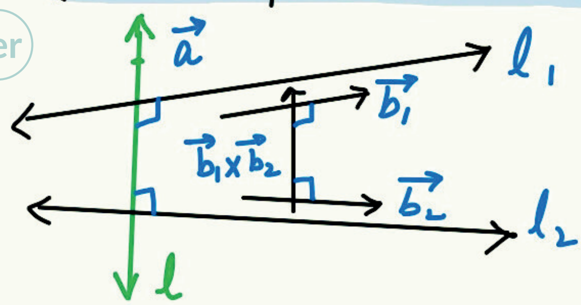
$$\vec{r} = \vec{a} + \lambda(\vec{b}_1 \times \vec{b}_2)$$



- Cartesian Forms are also called as Symmetric or Standard Forms
- Each of the forms can be easily converted to other.

3. Angle b/w 2 Lines

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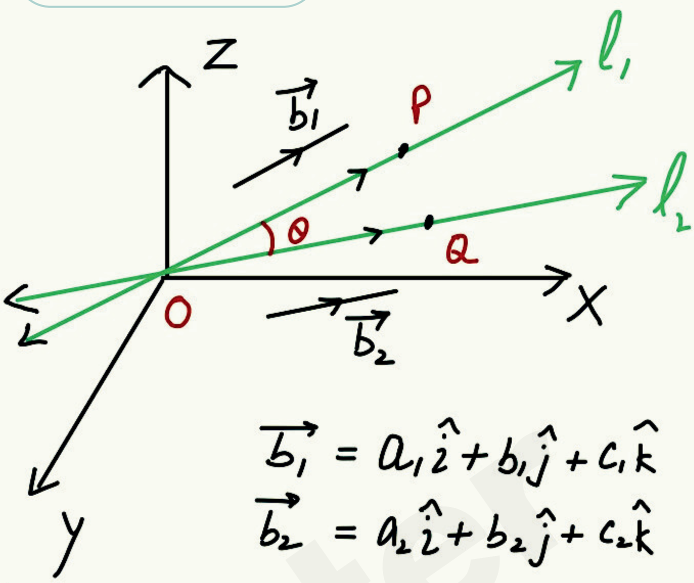
Two Given Lines:

$l_1: \ \vec{r} = \vec{a}_1 + \lambda \vec{b}_1$

$l_2: \ \vec{r} = \vec{a}_2 + \mu \vec{b}_2$

Angle b/w l_1 & l_2

$=$ **Angle b/w $\vec{b}_1$ & $\vec{b}_2$**



$\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$\vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

$$\therefore \quad \boxed{\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \right|} \qquad \left(\cdot \sin\theta = \sqrt{1-\cos^2\theta}\right)$$

or

$$\boxed{\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}} \right|}$$

or

$$\boxed{\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|}$$

KEY POINTS:

Maths Better

- If $l_1 \perp l_2$, $\theta = 90^\circ \Rightarrow \cos\theta = 0$, then $\vec{b}_1 \cdot \vec{b}_2 = 0$

  $\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$ Condition of Perpendicularity

- If $l_1 \parallel l_2$, $\theta = 0^\circ \Rightarrow \sin\theta = 0$, then $\vec{b}_1 \times \vec{b}_2 = \vec{0}$

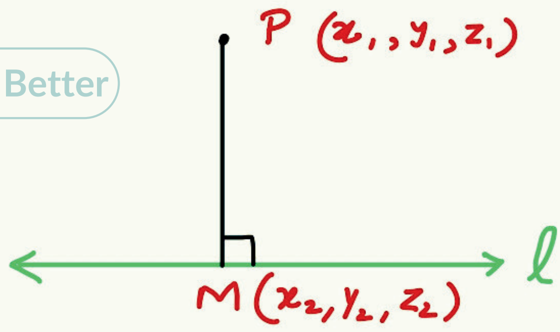
  or $\vec{b}_1 = \lambda \vec{b}_2 \Rightarrow \boxed{\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}}$ Condition of Parallelism

- If θ is acute, $\cos\theta$ is positive
- If θ is obtuse, $\cos\theta$ is negative

4. Foot of Perpendicular

Given :

$\vec{r} = \vec{a} + \lambda\vec{b}$ (Line ℓ)



- $P(x_1, y_1, z_1)$ is any point outside of ℓ
- $PM \perp \ell$; M is the foot of perpendicular from P

Algorithm

Steps to Find the Coordinates of M (Foot of $\perp$)

1. Convert Eq. of ℓ to parametric (if not)
2. Get the General Coordinates of any point on ℓ in terms of λ, μ etc.

3. General Coordinates will represent M as well.
4. Find D.R.'s of P M $(\vec{OM} - \vec{OP})$ i.e. $< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$
   (D.R.s of ℓ are given)
5. As $PM \perp \ell$, use $a_1a_2 + b_1b_2 + c_1c_2 = 0$
6. Get the value of λ (scalar) from step 5.
7. Put the value of λ in step 3 to get the Coordinates of M

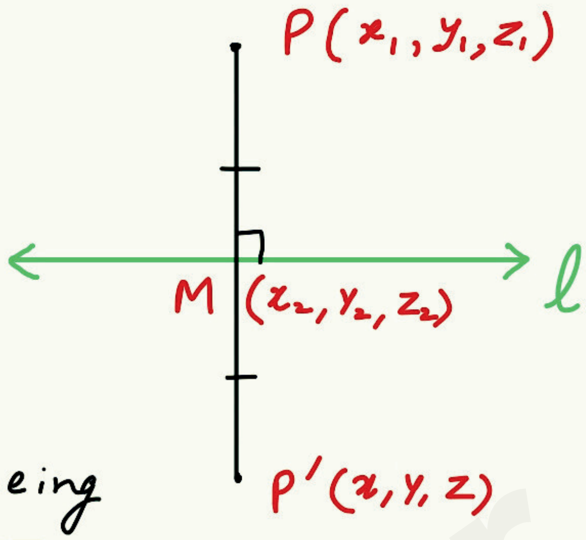
- If line ℓ is x, y or z-axis, Coordinates of M will be $(k, 0, 0)$, $(0, k, 0)$ or $(0, 0, k)$ resp.

5. Image of a Point

Given :

$\vec{r} = \vec{a} + \lambda \vec{b}$ (Line ℓ)

- Point $P(x_1, y_1, z_1)$ is any point outside of ℓ
- $PM \perp \ell$, $M(x_2, y_2, z_2)$ being the foot of perpendicular
- Let $P'(x, y, z)$ be the IMAGE of P



Algorithm

Steps to Find the Coordinates of P′ (Image of P)

1. Get the coordinates of M (as foot of $\perp$ PM)
   (Follow all the Steps 1 to 7 as in FOOT OF $\perp$)

8. As $PM = MP'$, Consider M as the mid-point of PP'

9. Get the coordinates of P', Using Section-Formula
   (i.e. Mid-point Formula)

\# To Find the distance between P and P′ (or M)
— Use Distance Formula

\# To Find the equation of line joining P and P′ (or M)
— Use Two-Point Form

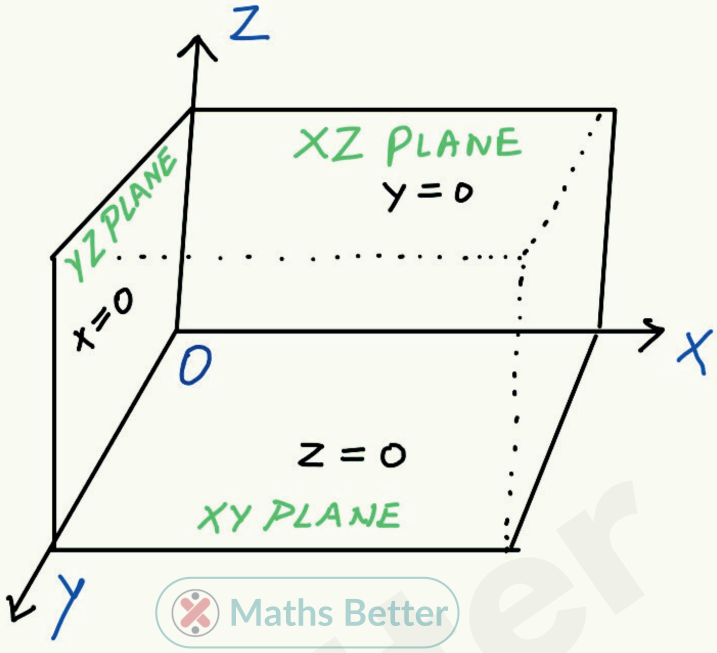
6. Coplanar & Skew Lines

COPLANAR LINES

- Lines in the **SAME Plane**

e.g. XY, YZ etc. are

- Either **Parallel**
- Or **Intersecting**
- Or **Coincident**



Neither Parallel Nor Intersecting Lines Are

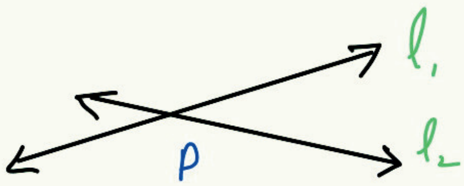
- **SKEW LINES** and
- **NON-COPLANAR**

Distance between two Lines:

Intersecting Lines: ZERO

(COPLANAR)

(TO FIND THE POINT OF INTERSECTION)

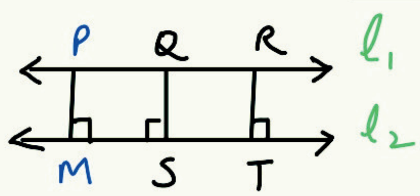


Parallel Lines: EQUIDISTANT

(COPLANAR)

↓

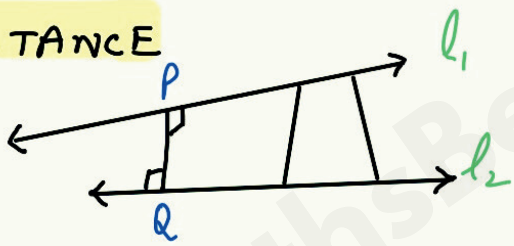
(⊥ DISTANCE OF ANY POINT ON ONE LINE FROM THE OTHER)



Skew Lines: SHORTEST DISTANCE

(NON-COPLANAR)

(LINE ⊥ TO BOTH THE LINES)

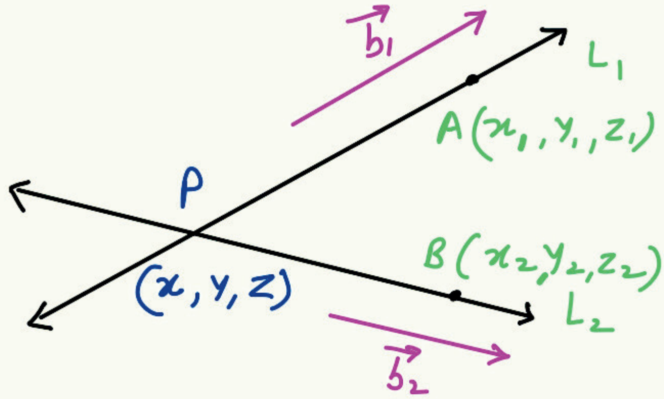


7. Point of Intersection

Given : Two Lines

$L_1: \vec{r} = \vec{a}_1 + \lambda \vec{b}_1$

$L_2: \vec{r} = \vec{a}_2 + \mu \vec{b}_2$



Cartesian + Parametric Forms:

$$L_1: \quad \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} = \lambda$$

$$L_2: \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} = \mu$$

Algorithm

Steps to Find the Coordinates of the Point of Intersection

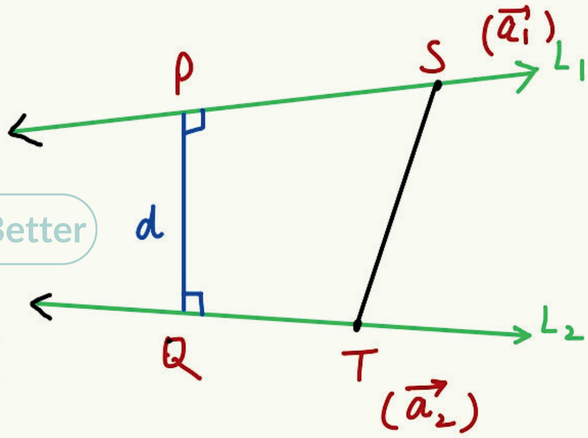
1. Get the coordinates of general points on both the lines in terms of λ and μ i.e. $x = x_1 + a_1\lambda$ etc. and $x = x_2 + a_2\mu$ etc.
2. Equate x, y and z so obtained in step 1.
3. Solve any two equations obtained in step 2 for λ and μ i.e. $x_1 + a_1\lambda = x_2 + a_2\mu$; $y_1 + b_1\lambda = y_2 + b_2\mu$ and $z_1 + c_1\lambda = z_2 + c_2\mu$
4. Put the values of λ and μ in 3rd equation. If they satisfy it $\Rightarrow$ Lines are intersecting, else not.
5. Put the values of λ and μ in step 1 to get the coordinates of the point of intersection.

8. Shortest Distance

Given : Two Lines

$L_1: \vec{r} = \vec{a}_1 + \lambda \vec{b}_1$

$L_2: \vec{r} = \vec{a}_2 + \mu \vec{b}_2$



Cartesian + Parametric Forms:

$$L_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} = \lambda$$

$$L_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} = \mu$$

SHORTEST DISTANCE BETWEEN SKEW LINES

PQ = Projection of ST along PQ

$$= \vec{ST} \cdot \hat{PQ} = (\vec{a}_2 - \vec{a}_1) \cdot \frac{\vec{PQ}}{|\vec{PQ}|}$$

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

DISTANCE BETWEEN PARALLEL LINES

If $L_1 \parallel L_2$, then $\vec{b}_1 = \vec{b}_2 = \vec{b}$

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \times \vec{b}}{|\vec{b}|} \right|$$

LENGTH OF ⊥ FROM POINT $(\vec{p})$ to $\vec{r} = \vec{a} + \lambda \vec{b}$

$$d = \left| \frac{(\vec{p} - \vec{a}) \times \vec{b}}{|\vec{b}|} \right|$$

Algorithm

Steps to Find the S. D. without using Formula

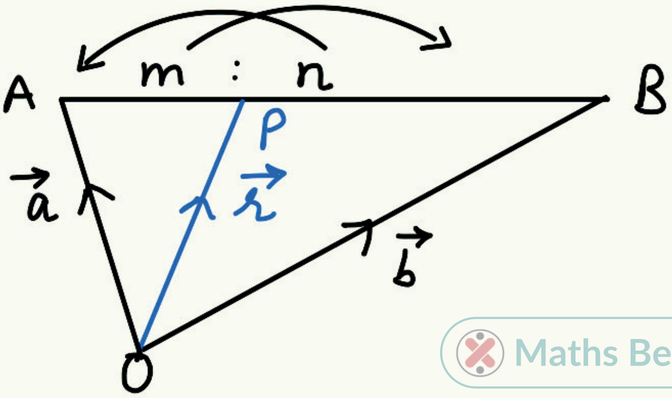
1. Get the Coordinates of P and Q in terms of λ and μ
2. Get the direction ratios of PQ (coordinates of (Q − P))
3. As $PQ \perp L_1$ and $PQ \perp L_2$, get values of λ and μ using Condition of perpendicularity (Dot product = zero)
4. Put values of λ and μ in step 1 to get P and Q
5. Get PQ using Distance Formula.

9. Section Formulae

Internal Division

$$\frac{AP}{PB} = \frac{m}{n}$$

(P is inside of AB)



$$\boxed{\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}}$$

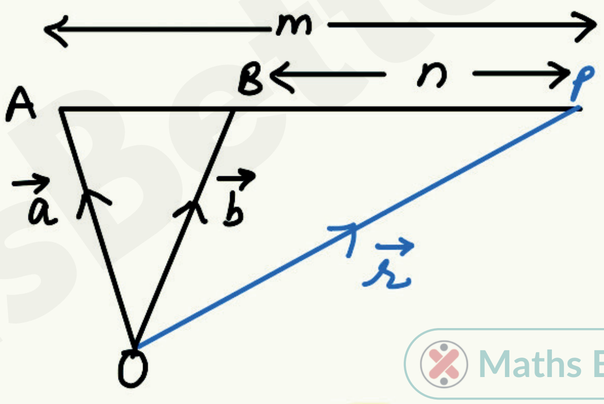
let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

then $P(x, y, z)$ is : $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$

External Division

$$\frac{AP}{PB} = \frac{m}{n}$$

(P is outside of AB)



$$\boxed{\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}}$$

(Replace n by −n)

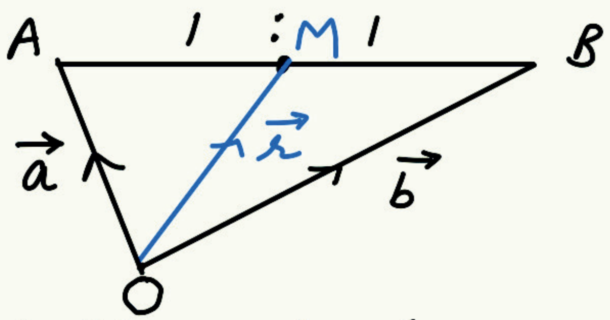
let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

then $P(x, y, z)$ is : $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

Mid-Point Formula

AM = MB

$$\vec{OM} = \boxed{\vec{r} = \frac{\vec{a} + \vec{b}}{2}}$$



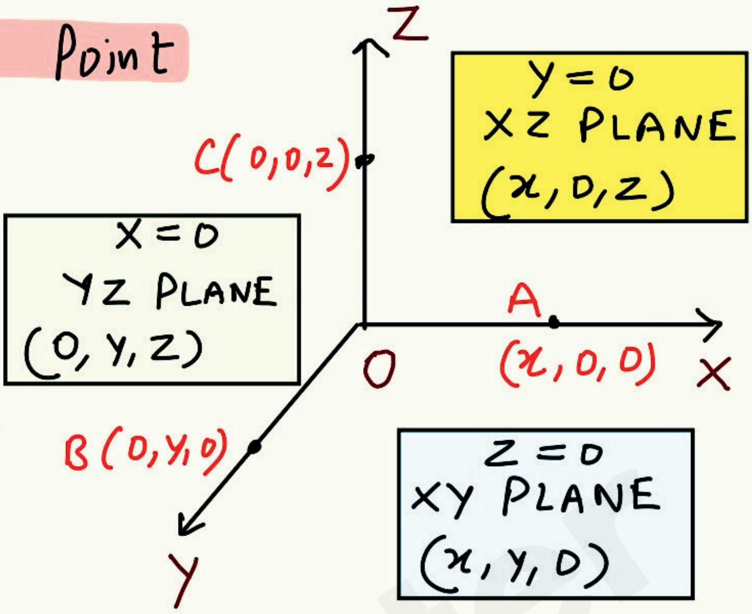
let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

then $M(x, y, z)$ is : $\left(\frac{x_1 + x_2}{m+n}, \frac{y_1 + y_2}{m+n}, \frac{z_1 + z_2}{m+n}\right)$

10. Miscellaneous

Coordinates of a Point

- On a plane :-
- On an axis :-



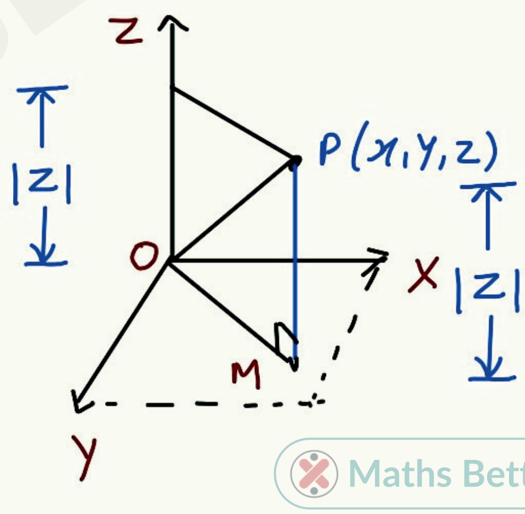
- X, Y, Z Axes divide the space into 8 octants

Perpendicular Distance

$\perp$ distance of (x, y, z) from yz-plane $= |x|$

– from zx-plane $= |y|$

– from xy-plane $= |z|$



D.C's of a line equally inclined to 3 axes :

$$\left\langle \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}} \right\rangle$$

Condition For Coplanarity of 2 Lines (Intersecting Lines)

$\boxed{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0}$ i.e. $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

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